

A Condition to prescribe Mean curvature Equations.

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Abstract

Let (B^n, g) the n -dimensional Ball. In this paper we show the existence of a solution to the prescribing mean curvature equation:

$$\left\{ \begin{array}{ll} \Delta_g u = 0 & \text{in } B^n, \\ \frac{\partial u}{\partial \eta} + \frac{(n-2)}{2}u = h \frac{(n-2)}{2}u^{n/n-2} & \text{over } \partial B^n, \end{array} \right. \quad (1)$$

In the case h is rotationally symmetric, is well known that the Kazdan-Warner condition implies that a necessary condition for (1) to have a solution is:

$$h > 0 \text{ somewhere and } h'(r) \text{ change signs}$$

This condition was improved to:

$$h'(r) \text{ changes signs where } h > 0 \quad (2)$$

However, is this a sufficient condition?. In this paper we prove that if $h(r)$ satisfies a flatness condition and is radially symmetric then (2) is the necessary and sufficient condition for (1) to have a solution on the subcritical level.

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1 Introduction

Lets take (B^n, g_0) , where $n \geq 3$ and g_0 is the euclidian metric that has a flat scalar curvature inside the ball and constant mean curvature $h_0 = 1$ on the boundary ∂B^n . A classic problem of differential geometry is the characterization of the pair of functions R y h , R inside the ball, and h in the boundary, such that there is a metric g , conformal to g_0 , with R as the prescribed scalar curvature on the ball, and h the prescribed mean curvature on ∂B^n .

Given the R and h functions, the existence of the metric g is equivalent to the existence of smooth function u which satisfies the nonlinear partial differential equation in the Sobolev critical exponent.

$$\left\{ \begin{array}{ll} \Delta_g u + \frac{(n-2)}{4(n-1)} R u^{\frac{n+2}{n-2}} = 0 & \text{on } B^n, \\ \frac{\partial u}{\partial \eta} + \frac{(n-2)}{2} u = h \frac{(n-2)}{2} u^{n/(n-2)} & \text{on } \partial B^n, \end{array} \right. \quad (3)$$

In the last two decades, this problem got a lot of attention and several sufficient conditions were found. (See the articles [2] [4] [6] [7] [10]) However the question remained on whether they were or not necessary conditions.

A open problem of common concern, is to think in a way to make the Kazdan-Warner condition sufficient and necessary. If h is rotationally symmetric, the Kazdan Warner condition is:

$$h'(r) \text{ change signs where } h > 0$$

In the paper [11] was proved that if R is flat and the function $h : \partial B^n \rightarrow \mathbb{R}$ is radially symmetric and satisfies:

$$(i) \ h(x) > 0 \text{ y } \frac{\partial h}{\partial r} \leq 0 \text{ si } |x| < 1$$

$$(ii) h(x) \leq 0 \text{ si } |x| \geq 1$$

Then the asociated diferencial equation system has no solution.

In this work we show that if $\frac{\partial h}{\partial r}$ change signs where h is positive and have a flatness condition, then is a sufficient condition to guarantee the existence of the metric g . Based on the ideas for the sphere from [3] , we reconstruct the priori estimatives over the functional related to the partial diferencial equation system for the ball and use a pass mountain scheme of critical points at infinity to guarantee the existence of solutions for the problem in the subcritical level and then prove the solutions are uniformly bounded at the critical level. The goal of this article is to prove the next result:

Theorem 1.1. *Let $n \geq 3$ and $h = h(r)$ a smooth function on ∂B^n simetric along x_n axis. Suppose that h has at least two positive local maximum and satisfies a flatness condition near every critical point τ_0 as follows:*

$h(r) = h(\tau_0) + a|r - r_0|^\alpha + k(|r - r_0|)$ con $a \neq 0$ y $n - 3 < \alpha < n - 1$. If $h'(r)$ change signs where $h > 0$ then the equation:

$$\left\{ \begin{array}{ll} \Delta_g u = 0 & \text{in } B^n, \\ \frac{\partial u}{\partial \eta} + \frac{(n-2)}{2}u = h \frac{(n-2)}{2}u^p & \text{in } \partial B^n, \end{array} \right. \quad (4)$$

where $1 < p \leq \frac{n}{n-2}$ and $k'(s) = o(s^{\alpha-1})$, have solution.

In this Proposition, we add a smooth-flatness condition in the h function (as defined in [14]) under the Kazdan-Warner type, also we ask that in each critical point of h (h has at least two maximum) the derivatives of h vanish until the $(n-3)$ and some of larger order ($< (n-1)$) remain diferent from zero. This extra consideration help us to prove some inequalities needed in the variational scheme, and adding the Kazdan-Warner type we reach a necessary and sufficient condition.

2 The Subcritical Case

In this section we construct a Maximini variational scheme to found the subcritical solution, first of all we rewrite the problem.

$$\begin{cases} \Delta_g u = 0 & \text{in } B^n, \\ \frac{\partial u}{\partial \eta} + \gamma_n u = h \gamma_n u^p & \text{in } \partial B^n \end{cases} \quad (5)$$

with $1 < p < \tau$, where $\gamma_n = \frac{n-2}{2}$, $\tau = \frac{n}{n-2}$ and $k'(s) = o(s^{\alpha-1})$, has a solution. Now lets consider the following functionals:

$$J_p(u) = \int_{\partial B^n} \gamma_n h u^{p+1} d\sigma$$

$$E(u) = \int_{B^n} |\nabla u|^2 dv + \int_{\partial B^n} \gamma_n u^2 d\sigma$$

Take $\|u\| = \sqrt{E(u)}$ the norm $H_1^2(B^n)$ and let:

$$S = \{u \in H_1^2(B^n) : E(u) = E(1) = \gamma_n |S^{n-1}|, u \geq 0\}$$

where $|S^{n-1}|$ is the volume of S^{n-1} .

Lemma 2.1. *A scalar multiple of a critical point of J_p in S is a solution of (5)*

The proof is a straight-forward calculation by Lagrange multipliers.

Now we want to show the needed inequalities and conditions in order to use the Pass Mountain Theorem in each local maximum of h , unlike the clasic Mountain Pass Theorem, we are going to use neighborhoods of critical points at infinity. Lets define some necessary tools for our work.

2.1 Previous Definitions

Lets consider $S^{n-1}(0)$ the n -dimensional sphere and obtain the stereographic projection over \mathbb{R}^{n-1} coming from the north pole into the $y = -1$ plane. The extension of this application to the ball is given as showed next.

Definition 2.2. Let (B^n, g_0) the n -dimensional euclidean ball and $\mathbb{R}_{-1}^n = \{x \in \mathbb{R}^n : x_n \leq -1\}$ lets define the function $\phi : B^n \longrightarrow \mathbb{R}_{-1}^n$ as the application:

$$(y, s) \longrightarrow \left(\frac{4y}{||y||^2 + (s-1)^2}, \frac{||y||^2 + (s+1)^2 - 4}{||y||^2 + (s-1)^2} \right)$$

and the function $\phi^{-1} : \mathbb{R}_{-1}^n \longrightarrow B^n$ given by:

$$(x, t) \longrightarrow \left(\frac{4x}{||x||^2 + (t-1)^2}, \frac{||x||^2 + (t+1)^2 - 4}{||x||^2 + (t-1)^2} \right)$$

The pullback metric asociated with the metric g_0 and ϕ is:

$$\phi^*(g_0) = \frac{16\delta_{ij}}{(||x||^2 + (t-1)^2)^2}.$$

Consider now the coordinate system over R^n with the south pole of the ball as the coordinate origin. Let g be the usual metric over B^n , and the dilation $T : R_{-1}^n \longrightarrow R_{-1}^n$ such as $T(x, t) = (\beta x, \beta(t+1) - 1)$ and $\phi : (B^n, g) \longrightarrow (R_{-1}^n, \delta_{ij})$ the extension of the stereographic projection to the ball from the north pole, next we find a family of positive solutions u to the system of partial diferential equations

$$\begin{cases} \Delta_g u = 0 & \text{en } B^n, \\ \frac{\partial u}{\partial \eta} + \gamma_n u = \gamma_n u^p & \text{en } \partial B^n \end{cases} \quad (6)$$

With $p < \tau$. As it can be seen, this solutions solve the problem of prescribing zero scalar curvature and mean curvature $h = 1$ in the ball. To construct the solutions lets check the pullback.

$$(\phi \circ T)^*(g) = T^*(\phi^*(g)) = \phi^*(g)_{T(z')} \langle dT(e_i), dT(e_j) \rangle = \frac{16\beta^2\delta_{ij}}{(||\beta x||^2 + (\beta(t+1) - 2)^2)^2}.$$

since $(\phi \circ T)^*(g)$ and $\phi^*(g)$ are conformal there is a positive smooth function u such as:

$$u^{\frac{4}{n-2}} \frac{16\delta_{ij}}{(\|x\|^2 + (t-1)^2)^2} = \frac{16\beta^2\delta_{ij}}{(\|\beta x\|^2 + (\beta(t+1) - 2)^2)^2}$$

and

$$u^{\frac{2}{n-2}} = \frac{\beta(\|x\|^2 + (t-1)^2)}{(\|\beta x\|^2 + (\beta(t+1) - 2)^2)}$$

So we have found a infinite family of solutions for (6). Now let $\phi : B^n \rightarrow \mathbb{R}_{-1}^n$ the extension of the stereographic projection to the ball from north pole, and let $\phi(z) = z'$, with $z = (y, s)$ and $z' = (x, t)$, then it holds that:

$$\|z'\|^2 = \frac{4\|z\|^2}{\|z\|^2 - 4s} \quad (7)$$

Setting $z' = (x, t)$ and $z = (y, s)$ in our family of solutions we have

$$u^{\frac{2}{n-2}} = \frac{\beta^2(\|z'\|^2 - 4t)}{(\beta^2\|z'\|^2 + 4 - 4\beta(t+1))}$$

Using (7) and $t = \frac{\|y\|^2 + (s+1)^2 - 4}{\|y\|^2 + (s-1)^2} = \frac{\|z\|^2 - 4}{\|z\|^2 - 4s}$ replacing those values on the solution it follows that:

$$u(z) = \left(\frac{4\beta}{(\beta-1)^2\|z\|^2 + 4s(\beta-1) + 4\beta} \right)^{\frac{n-2}{2}}$$

Its necessary to make several estimates on the boundary of the ball $\partial B^n = S^{n-1}$, so it is very useful if we have a reduced form of $u(z)$ there, take the $t = -1$ plane, then:

$$-1 = \frac{\|y\|^2 + (s+1)^2 - 4}{\|y\|^2 + (s-1)^2} = \frac{\|z\|^2 - 4}{\|z\|^2 - 4s}$$

So if $z \in S^{n-1}$

$$u = \left(\frac{\beta}{\frac{(\beta^2-1)\|z\|^2}{4} + 1} \right)^{\frac{n-2}{2}}$$

Setting coordinates on S^{n-1} as follows, if $z \in S^{n-1}$, $z = (r, \theta)$ with $0 \leq r \leq \pi$ and $\theta \in S^{n-1}$, then $\frac{\|z\|}{2} = \text{sen} \frac{r}{2}$, this means:

$$u(z) = \left(\frac{\beta}{(\beta^2 - 1)\text{sen}^2 \frac{r}{2} + 1} \right)^{\frac{n-2}{2}}$$

with $1 \leq \beta \leq \infty$, if $\lambda = \frac{1}{\beta}$ then $0 \leq \lambda \leq 1$ and :

$$u(z) = \left(\frac{\lambda}{(1 - \lambda^2)\text{sen}^2 \frac{r}{2} + 1} \right)^{\frac{n-2}{2}} = \left(\frac{\lambda}{(1 - \lambda^2)\frac{z_n}{2} + \lambda^2} \right)^{\frac{n-2}{2}}$$

Now lets introduce a result that will be helpful to see the solutions we have found belong on the space S .

Proposition 2.3. *The volume of S^{n-1} can be seen as:*

$$|S^{n-1}| = |S^{n-2}| \int_0^\pi \sin^{n-2}(t) dt$$

Proof. The demonstration of this is a straight-forward calculation, from the volume of S^{n-1} using n -dimensional spherical coordinates.

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Now let \tilde{q} a point in S^{n-1} the family of solutions u is simetric along the axis given by e_n y \tilde{q} . Lets call $u_{\tilde{q}, \lambda} = u_q$ the solutions with mass center given by:

$$q = q(u) = \frac{\int_{B^n} z u^{2\tau}(z) dv}{\int_{B^n} u^{2\tau}(z) dv}$$

Lets observe that the point q is determined by \tilde{q} y λ and is defined on the axis given by e_n and \tilde{q} . For the rest of our work lets suposse unless stated otherwise that \tilde{q} is the coordinate origin.

Proposition 2.4. *The family of solutions u_q belong to S*

Proof.

Since u_q holds (5) then:

$$\begin{cases} \Delta_g u_q = 0 & \text{en } B^n, \\ \frac{\partial u_q}{\partial \eta} + \gamma_n u_q = \gamma_n u_q^p & \text{en } \partial B^n \end{cases} \quad (8)$$

Integrating by parts we reach:

$$E(u_q) = \int_{S^{n-1}} \gamma_n u_q^{p+1} d\sigma$$

In order to prove $u_q \in S$ is only left to see that $\int_{S^{n-1}} \gamma_n u_q^{p+1} d\sigma = |S^{n-1}|$, lets take n -dimensional spherical coordinates $d\sigma = \sin^{n-2} r \sin^{n-1} \theta_1 \dots \sin \theta_{n-2} dr d\theta_1 \dots d\theta_{n-2}$ and u_q restricted to S^{n-1} then:

$$\begin{aligned} \int_{S^{n-1}} u_q^{p+1} d\sigma &= \int_{S^{n-1}} \left(\left(\frac{\lambda}{\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2}} \right)^{\frac{n-2}{2}} \right)^{\frac{2(n-1)}{n-2}} d\sigma \\ &= \int_{S^{n-1}} \left(\frac{\lambda}{\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2}} \right)^{n-1} d\sigma = \int_0^{2\pi} d\theta_{n-2} \int_0^\pi \sin \theta_{n-3} d\theta_{n-3} \dots \int_0^\pi \frac{\lambda^{n-1} \sin^{n-2} r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \\ &= |S^{n-2}| \int_0^\pi \frac{\lambda^{n-1} \sin^{n-2} r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \end{aligned}$$

Now if $r = 2 \tan^{-1}(\lambda \tan \frac{u}{2})$ then:

$$E(u_q) = \gamma_n \int_{S^{n-1}} u_q^{p+1} d\sigma = |S^{n-2}| \int_0^\pi \sin^{n-2} u du = \gamma_n |S^{n-1}|$$

■

Lets consider $\varphi : B^n \longrightarrow B^n$ a dilation from the ball on the ball and $T_\varphi : H_1^2(B^n) \longrightarrow H_1^2(B^n)$ given by:

$$T_\varphi(u(x)) = u(\varphi(x)) \cdot [\det(d\varphi(x))]^{\frac{n-2}{2(n-1)}}$$

This family of conformal transformations leaves the equation (6), the energy $E(\cdot)$ and the functional J_p invariant as proved in [10]. As consequence we have the next result.

Lemma 2.5. *If u is a solution of (5) then $T_\varphi u$ is a solution too.*

2.2 A Suitable Neighborhood

Definition 2.6. Let $\Sigma = \left\{ u \in S : |q(u)| \leq \rho_0, \|v\| = \min_{t,q} \|u - tu_q\| \leq \rho_0, t \in \mathbb{R} \right\}$

the set of solutions with mass center q near the south pole O . Let's denote $\overset{\circ}{\Sigma}$ the inside of Σ .

On this set, u has the mass center in the same neighborhood that u_q that solves (6). This is the neighborhood of critical points on infinity corresponding to the south pole O . Let's check now some necessary results in the development of the inequalities of the functional.

Lemma 2.7. Let $u \in \Sigma$ and $v = u - t_0 u_q$ y T_φ as previously defined. Then:

$$\int_{S^{n-1}} T_\varphi v \, dv = 0 \quad \text{and} \quad \int_{S^{n-1}} x_i T_\varphi v \, dv = 0.$$

Where x_i represent the coordinate functions.

This result is obtained in [3].

Lemma 2.8. Let $u \in \Sigma$, $v = u - t_0 u_q$ and $T_\varphi : H_1^2(B^n) \longrightarrow H_1^2(B^n)$ as previously defined. We have the following equalities:

$$\langle T_\varphi v, K \rangle_{H_1^2(B^n)} = 0 \quad \text{and} \quad \langle T_\varphi v, x_i \rangle_{H_1^2(B^n)} = 0$$

with x_i the coordinate functions and K a constant function.

Proof. A straight forward calculation, using the lemma (2.7) for some parts.

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3 Estimates on J_p

We will show estimates on J_p near the south pole (O, θ) where we assume a positive local maximum, the estimates near another local maximum will be the same. Now by the hypothesis in (1.1) then $h(r) = h(0) - ar^\alpha$ for some $a > 0$, $n - 3 < \alpha < n - 1$ in a open set that has the coordinate origin O .

Proposition 3.1. *For all $\delta_1 > 0$ exists $P_1 \leq \tau$ such that for all $P_1 \leq P \leq \tau$ holds:*

$$\sup_{\overset{\circ}{\Sigma}} J_p(u) > h(0)|S^{n-1}| - \delta_1$$

Proof. Lets prove $J_\tau(u_{\lambda,O}) \rightarrow h(0)|S^{n-1}|$ when $\lambda \rightarrow 0$. Lets take a fixed number $\tau_0 \neq 0$ nearly enough to zero then:

$$\begin{aligned} J_\tau(u_{\lambda,O}) &= \int_{S^{n-1}} h(r) u_{\lambda,O}^{\tau+1} d\sigma = \int_{S^{n-1}} h(r) \left(\frac{\lambda}{\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2}} \right)^{n-1} d\sigma \\ &= |S^{n-2}| \int_0^\pi \frac{\lambda^{n-1} h(r) \sin^{n-2} r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \\ &= |S^{n-2}| \left\{ \int_0^{\tau_0} \frac{\lambda^{n-1} h(r) \sin^{n-2} r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr + \int_{\tau_0}^\pi \frac{\lambda^{n-1} h(r) \sin^{n-2} r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \right\} \\ &= |S^{n-2}| \left\{ \int_0^{\tau_0} \frac{\lambda^{n-1} (h(0) - ar^\alpha) \sin^{n-2} r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr + \int_{\tau_0}^\pi \frac{\lambda^{n-1} h(r) \sin^{n-2} r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \right\} \end{aligned}$$

adding and subtracting:

$$\int_{\tau_0}^\pi \frac{\lambda^{n-1} h(0) \sin^{n-2} r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr$$

we get:

$$= h(0)|S^{n-1}| + |S^{n-2}| \left\{ \int_0^{\tau_0} \frac{-ar^\alpha \lambda^{n-1} \sin^{n-2} r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr + \int_{\tau_0}^\pi \frac{\lambda^{n-1} (h(r) - h(0)) \sin^{n-2} r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \right\}$$

Lets call:

$$I_1 = \left| \int_0^{\tau_0} \frac{r^\alpha \lambda^{n-1} \sin^{n-2} r}{(\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2})^{n-1}} dr \right|$$

$$I_2 = \left| \int_{\tau_0}^{\pi} \frac{\lambda^{n-1}(h(r) - h(0)) \text{sen}^{n-2} r}{(\lambda^2 \cos^2 \frac{r}{2} + \text{sen}^2 \frac{r}{2})^{n-1}} dr \right|$$

We will show $I_1 \rightarrow 0$ and $I_2 \rightarrow 0$ when $\lambda \rightarrow 0$, clearly:

$$\begin{aligned} I_1 &= \left| \int_0^{\tau_0} \frac{r^\alpha \lambda^{n-1} (2 \text{sen} \frac{r}{2} \cos \frac{r}{2})^{n-2}}{(\lambda^2 \cos^2 \frac{r}{2} + \text{sen}^2 \frac{r}{2})^{n-1}} dr \right| \leq \lambda 2^{n-2} \left| \int_0^{\tau_0} \frac{r^\alpha (\text{sen} \frac{r}{2})^{n-2} (\lambda^2 \cos^2 \frac{r}{2} + \text{sen}^2 \frac{r}{2})^{\frac{n-2}{2}}}{(\lambda^2 \cos^2 \frac{r}{2} + \text{sen}^2 \frac{r}{2})^{n-1}} dr \right| \\ &\leq \lambda 2^n \left| \int_0^{\tau_0} \frac{\left(\frac{r}{2}\right)^2 r^{\alpha-2} (\text{sen} \frac{r}{2})^{n-2}}{(\text{sen}^2 \frac{r}{2})^{n/2}} dr \right| \leq \lambda 2^n \left| \int_0^{\tau_0} \frac{\left(\frac{r}{2}\right)^2 r^{\alpha-2}}{(\text{sen}^2 \frac{r}{2})^2} dr \right| \leq \lambda 2^n \int_0^{\tau_0} r^{\alpha-2} dr = c_1 \lambda \frac{r_0^{\alpha-1}}{\alpha-1} \end{aligned}$$

A straight-forward calculation also shows that.

$$I_2 = \left| \int_{\tau_0}^{\pi} \frac{\lambda^{n-1}(h(r) - h(0))(2 \text{sen} \frac{r}{2} \cos \frac{r}{2})^{n-2}}{(\lambda^2 \cos^2 \frac{r}{2} + \text{sen}^2 \frac{r}{2})^{n-1}} dr \right| \leq C_2 \lambda (\pi - \tau_0)$$

Hence if $\lambda \rightarrow 0$ $I_2 \rightarrow 0$ and $J_\tau(u_{\lambda,O}) \rightarrow h(0)|S^{n-1}|$. Then if we take $\delta_1 > 0$, we can choose λ_0 such as $u_{\lambda,O} \in \overset{\circ}{\Sigma}$ and:

$$J_\tau(u_{\lambda,O}) > h(0)|S^{n-1}| - \frac{\delta_1}{2}$$

Since J_p is continuous respect p , given a fixed $u_{\lambda_0,O}$ there is P_1 such as for all $P \geq P_1$:

$$\sup_{\overset{\circ}{\Sigma}} J_p(u) > h(0)|S^{n-1}| - \delta_1$$

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Similarly to Lemma 2.1 in [3] we can show that.

Lemma 3.2. *Let $\epsilon > 0$ and $\bar{B}_\epsilon(0) = \{x \in S^{n-1}(e_n), |x| \leq \epsilon\}$. for $n-3 < \alpha < n-1$, p nearly enough to τ and for λ and $|\tilde{q}|$ small enough we have:*

$$J_p(u_{\lambda,\tilde{q}}) \leq (h(0) - C_1 |\tilde{q}|^\alpha) |S^{n-1}| (1 + O_p(1)) - C_1 \lambda^{\alpha+\delta_p}$$

Where $\delta_p > \tau - p$ and $O_p(1) \rightarrow 0$ when $p \rightarrow \tau$.

Lemma 3.3. *If $u \in \Sigma$ and $v = u - t_0 u_q$ as defined in (2.6) then u_q and v are orthogonal and hold:*

$$\int_{S^{n-1}} u_q^\tau v d\sigma = 0$$

Proof. As $E(u - t u_q)$ reach a minimum in $v = u - t_0 u_q$, on Σ by the definition of Σ we have:

$$\begin{aligned} 0 &= E'(u - t u_q)|_{t=t_0} = \int_{B^n} -2\nabla u \nabla u_q + 2t_0 |\nabla u_q|^2 dv + \int_{S^{n-1}} (-2\gamma_n u u_q + 2\gamma_n t_0 u_q^2) d\sigma \\ &= \int_{B^n} \nabla u \nabla u_q dv + \int_{S^{n-1}} \gamma_n u u_q d\sigma - t_0 \left(\int_{B^n} |\nabla u_q|^2 dv + \gamma_n \int_{S^{n-1}} u_q^2 d\sigma \right) \\ 0 &= \langle u, u_q \rangle - t_0 \langle u_q, u_q \rangle = \langle u - t_0 u_q, u_q \rangle \end{aligned}$$

And u_q y v are orthogonal in $H_1^2(B^n)$. Now lets take v as test function in the equation (6) and integrate by parts, then.

$$\int_{B^n} -v \Delta u_q dv = 0$$

we reach.

$$\int_{B^n} \nabla u \nabla u_q dv + \int_{S^{n-1}} \gamma_n u_q v d\sigma = \int_{S^{n-1}} \gamma_n u_q^\tau v d\sigma$$

And so:

$$0 = \langle u_q, v \rangle_{H_1^2(B^n)} = \int_{S^{n-1}} \gamma_n u_q^\tau v d\sigma$$

As we wanted to show.

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Lemma 3.4. *(About the mass center)*

1. *Let q , λ and \tilde{q} as previously. Then if q small enough:*

$$|q|^2 \leq C (|\tilde{q}|^2 + \lambda^4)$$

2. Let ρ_o and v defined in (2.6). then for ρ_0 small enough:

$$\rho_0 \leq |q| + C||v||$$

Proof.

1. Lets see if λ small enough.

$$|q - \tilde{q}| \leq C\lambda^2.$$

In this case the mass center of both the sphere S^{n-1} and the ball B^n is the same, if $\tilde{g} = u^{\frac{4}{n-2}}g$ then $d\tilde{v} = u^{\frac{2n}{n-2}}dv$ in B^n and $d\tilde{\sigma} = u^{\frac{2(n-1)}{n-2}}d\sigma$ in S^{n-1}

Now as $d\tilde{v} = u^{2\tau}dv$ we have:

$$\int_{B^n} x d\tilde{v} = \int_{B^n} x u^{2\tau} dv \quad \text{y} \quad \int_{B^n} d\tilde{v} = \int_{B^n} u^{2\tau} dv.$$

Then the mass center q holds:

$$q = \frac{\int_{B^n} x u^{2\tau} dv}{\int_{B^n} u^{2\tau} dv} = \frac{\int_{B^n} x d\tilde{v}}{\int_{B^n} d\tilde{v}} = \frac{\int_{S^{n-1}} x d\tilde{\sigma}}{\int_{S^{n-1}} d\tilde{\sigma}} = \frac{\int_{S^{n-1}} x u^{\tau+1} d\sigma}{\int_{S^{n-1}} u^{\tau+1} d\sigma}$$

Now as $\tilde{q} = O$, we can prove by means of a direct calculation that $|q_i| = 0$ for $i = 1 \dots n-1$. It can be seen that $|q_n| \leq C\lambda^2$ for some $C > 0$ because:

$$\begin{aligned} |q_n| &= \frac{1}{|S^{n-1}|} \int_{S^{n-1}} x_n u_q^{\tau+1} d\sigma = C \int_0^\pi (1 - \cos r) \left(\frac{\lambda}{\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2}} \right)^{n-1} \sin^{n-2} r \, dr \\ &= C \int_0^\pi \frac{1 - \cos^2 r}{1 + \cos r} \left(\frac{\lambda}{\lambda^2 \cos^2 \frac{r}{2} + \sin^2 \frac{r}{2}} \right)^{n-1} \sin^{n-2} r \, dr \leq C \lambda^{n-1} \int_0^\pi \frac{\tan^{n-2} \frac{r}{2}}{(\lambda^2 + \tan^2 \frac{r}{2})^{n-1}} \, dr \end{aligned}$$

Let $u = \tan \frac{r}{2}$, then:

$$\begin{aligned}
|q_n| &\leq C\lambda^{n-1} \int_0^\infty \frac{2u^n}{(\lambda^2 + u^2)^{n-1}} \frac{1}{1+u^2} du \leq C\lambda^{n-1} \int_0^\infty \frac{2u^n}{(\lambda^2 + u^2)^{n-1}} du \\
&\leq C\lambda^{n-1} \int_0^\infty \frac{2u}{(\lambda^2 + u^2)^{\frac{n-1}{2}}} du \leq C\lambda^{n-1} \frac{2}{3-n} (\lambda^2 + u^2)^{\frac{-n+3}{2}} \Big|_0^\infty \leq C\lambda^2
\end{aligned}$$

On the other hand, as:

$$|q|^2 \leq |q - \tilde{q}|^2 + |\tilde{q}|^2 + 2|q - \tilde{q}||\tilde{q}|$$

and

$$2|q - \tilde{q}||\tilde{q}| \leq |\tilde{q}|^2 + |q - \tilde{q}|^2$$

we have:

$$|q|^2 \leq 2|q - \tilde{q}|^2 + 2|\tilde{q}|^2 \leq C(|\tilde{q}|^2 + \lambda^4)$$

As we wanted to prove.

2. Consider $u = v + tu_q \in \partial\Sigma$ where

$$\Sigma = \left\{ u \in S : |q(u)| \leq \rho_0, \|v\| = \min_{t,q} \|u - tu_q\| \leq \rho_0, t \in \mathbb{R} \right\}$$

We are in $\partial\Sigma$ so $\|v\| = \rho_0$ or $|q(u)| = \rho_0$. if $\|v\| = \rho_0$ then we are done, if $|q(u)| = \rho_0$ then:

$$|q(u)| = \rho_0 = \left| \frac{\int_{B^n} x(v + tu_q)^{\tau+1} dv}{\int_{S^{n-1}} (v + tu_q)^{\tau+1} dv} \right|$$

as $\|v\| \leq \rho_0$ and ρ_0 are small, we take the taylor expansion of $\int_{S^{n-1}} x(v + tu_q)^{\tau+1} dv$ as:

$$t^{\tau+1} \int_{S^{n-1}} x u_q^{\tau+1} dv + (\tau+1)t^\tau \int_{S^{n-1}} x v u_q^\tau dv + O(\|v\|^2)$$

and $\int_{S^{n-1}} (v + tu_q)^{\tau+1} dv$ as :

$$t^{\tau+1} \int_{S^{n-1}} u_q^{\tau+1} dv + (\tau+1)t^\tau \int_{S^{n-1}} v u_q^\tau dv + O(\|v\|^2)$$

Now:

$$\rho_0 - |q| \leq \left| \frac{\int_{S^{n-1}} x(v + t u_q)^{\tau+1} dv}{\int_{S^{n-1}} (v + t u_q)^{\tau+1} dv} - \frac{\int_{S^{n-1}} x u_q^{\tau+1} dv}{\int_{S^{n-1}} u_q^{\tau+1} dv} \right|$$

Using the expansions and subtracting:

$$\begin{aligned} &= \left| \frac{(\tau+1)t^\tau \left(\int_{S^{n-1}} u_q^{\tau+1} dv \int_{S^{n-1}} x v u_q^\tau dv - \int_{S^{n-1}} x u_q^{\tau+1} dv \int_{S^{n-1}} v u_q^\tau dv \right) + O(\|v\|^2)}{\int_{S^{n-1}} u_q^{\tau+1} dv \int_{S^{n-1}} (v + t u_q)^{\tau+1} dv} \right| \\ &\leq \frac{(\tau+1)t^\tau \left(|S^{n-1}| \left| \int_{S^{n-1}} x v u_q^\tau dv \right| + \left| \int_{S^{n-1}} x u_q^{\tau+1} dv \right| \left| \int_{S^{n-1}} v u_q^\tau dv \right| \right) + O(\|v\|^2)}{|S^{n-1}| \left(t^{\tau+1} \int_{S^{n-1}} u_q^{\tau+1} dv + (\tau+1)t^\tau \int_{S^{n-1}} v u_q^\tau dv \right)} \end{aligned}$$

And as $\int_{S^{n-1}} u_q^{\tau+1} dv = |S^{n-1}|$ and $\int_{S^{n-1}} v u_q^\tau dv = 0$ we have:

$$\begin{aligned} \rho_0 - |q| &\leq \frac{(\tau+1)t^\tau |S^{n-1}|}{t^{\tau+1} |S^{n-1}|^2} \int_{S^{n-1}} |x| |v u_q^\tau| dv + O(\|v\|^2) \\ &= \frac{(\tau+1)}{t |S^{n-1}|} \int_{S^{n-1}} |v(\varphi(x)) u_q^\tau(\varphi(x)) \det[d(\varphi(x))]| dv + O(\|v\|^2) \\ &= \frac{(\tau+1)}{t |S^{n-1}|} \int_{S^{n-1}} \left| \frac{(T_\varphi u_q)^\tau}{\det[d(\varphi(x))]^{\frac{n-2}{2(n-1)}\tau}} \frac{T_\varphi v}{\det[d(\varphi(x))]^{\frac{n-2}{2(n-1)}}} \det[d(\varphi(x))] dv + O(\|v\|^2) \right| \\ &= \frac{(\tau+1)}{t |S^{n-1}|} \int_{S^{n-1}} |(T_\varphi u_q)^\tau T_\varphi v| dv \leq \frac{(\tau+1)}{t |S^{n-1}|} \int_{S^{n-1}} |T_\varphi v| dv \leq \frac{(\tau+1)}{t |B^n|} \left(\int_{B^n} (T_\varphi v)^2 dv \right)^{1/2} \\ &\leq \frac{(\tau+1)}{t |S^{n-1}|} \left(\int_{B^n} |\nabla T_\varphi v|^2 dv + \int_{S^{n-1}} (T_\varphi v)^2 d\sigma \right)^{1/2} + O(\|v\|^2) \leq \frac{(\tau+1)}{t |S^{n-1}|} \|v\| + O(\|v\|^2) \leq C \|v\| \end{aligned}$$

Because t is near 1 and $||v||$ is small, this show what we wanted to prove.

■

Now we show that J_p is bounded over the boundary of Σ .

Proposition 3.5. *There are some positive constants ρ_0, p_0, δ_0 such as for all $P \geq p_0$ and $u \in \partial\Sigma$ it holds:*

$$J_p(u) \leq h(0)|S^{n-1}| - \delta_0$$

Proof.

Lets take.

$$\bar{h}(x) = \begin{cases} h(x) & \text{in } B_{2\rho_0(0)}, \\ m & \text{in } S^{n-1}|B_{2\rho_0(0)} \end{cases} \quad (9)$$

Where $m = h|\partial B_{2\rho_0(0)}|$. Now lets define:

$$\bar{J}_p(u) = \int_{S^{n-1}} \bar{h}(x) u^{p+1} d\sigma$$

The estimates will be divided on two steps, in the step one we use the inequality.

$$|J_p(u) - \bar{J}_\tau(u)| \leq |\bar{J}_p(u) - \bar{J}_\tau(u)| + |J_p(u) - \bar{J}_p(u)|$$

To show the difference between $J_p(u)$ and $\bar{J}_\tau(u)$ is small. Step two will carry estimatives on $\bar{J}_\tau(u)$

Step 1. As in [3] it can be clearly seen that:

$$\bar{J}_p(u) \leq \bar{J}_\tau(u)(1 + o_p(1)) \quad (10)$$

Where $o_p(1) \longrightarrow 0$ when $p \longrightarrow \tau$.

Now lets check the diference between $J_p(u)$ and $\bar{J}_p(u)$.

$$|J_p(u) - \bar{J}_p(u)| = \int_{S^{n-1}|B_{2\rho_0}(0)} |h(x) - m| u^{p+1} d\sigma \quad (11)$$

$$\begin{aligned} &\leq C_1 \int_{S^{n-1}|B_{2\rho_0}(0)} u^{p+1} d\sigma \leq C_2 \int_{S^{n-1}|B_{2\rho_0}(0)} (t_0 u_q)^{p+1} d\sigma + C_2 \int_{S^{n-1}|B_{2\rho_0}(0)} v^{p+1} d\sigma \\ &\leq C_3 \lambda^{(n-1) - \frac{n-2}{2}\delta_p} + C_2 \int_{S^{n-1}|B_{2\rho_0}(0)} v^{p+1} d\sigma \leq C_3 \lambda^{(n-1) - \frac{n-2}{2}\delta_p} + C \|v\|^{p+1} \end{aligned}$$

last inequality as using the Beckner-Escobar Sobolev inequality we have.

$$\left(\int_{S^{n-1}|B_{2\rho_0}(0)} v^{p+1} d\sigma \right)^{\frac{1}{p+1}} \leq \left(\int_{S^{n-1}} v^{p+1} d\sigma \right)^{\frac{1}{p+1}} \leq C \left(\int_{B^n} |\nabla v|^2 dv + \int_{\partial B^n} \gamma_n v^2 d\sigma \right)^{1/2} \leq C \|v\|.$$

Now, as $\lambda^{(n-1) - \frac{n-2}{2}\delta_p}$ and $\|v\|^{p+1}$ are small. Using (10) and (11), the difference between $J_p(u)$ and $\bar{J}_p(u)$ is small.

Step 2.

Now we will carry out the estimatives on $J_\tau(u)$.

Let $u = v + t_0 u_q \in \partial\Sigma$. From (3.3) we have v and u_q are orthogonal respect the inner product asociated to $E(\cdot)$, meaning:

$$0 = \int_{B^n} (\nabla(u - t_0 u_q)) \cdot \nabla u_q dv + \gamma_n \int_{S^{n-1}} (u - t_0 u_q) u_q d\sigma$$

then:

$$t_0 E(u_q) = \int_{B^n} \nabla u \nabla u_q dv + \gamma_n \int_{S^{n-1}} u u_q d\sigma$$

Now as:

$$\|v\| = E(u - t_0 u_q) = E(u) + t_0^2 E(u_q) - 2t_0 \left(\int_{B^n} \nabla u \nabla u_q dv + \gamma_n \int_{S^{n-1}} u u_q d\sigma \right)$$

$$= E(u) + t_0^2 E(u_q) - 2t_0^2 E(u_q) = E(u) - t_0^2 E(u_q)$$

Moreover $E(u) = E(u_q) = \gamma_n |S^{n-1}|$ then.

$$||v||^2 = (1 - t_0^2) \gamma_n |S^{n-1}|$$

and

$$t_0^2 = 1 - \frac{||v||^2}{\gamma_n |S^{n-1}|}$$

Now:

$$\begin{aligned} \bar{J}_\tau(u) &= \int_{S^{n-1}} \bar{h}(x) u^{\tau+1} d\sigma \\ &\leq t_0^{\tau+1} \int_{S^{n-1}} \bar{h}(x) u_q^{\tau+1} d\sigma + (\tau+1) \int_{S^{n-1}} \bar{h}(x) u_q^\tau v d\sigma + \frac{\tau(\tau+1)}{2} \int_{S^{n-1}} \bar{h}(x) u_q^{\tau-1} v^2 d\sigma + O(||v||^2) \\ &= I_1 + (\tau+1) I_2 + \frac{\tau(\tau+1)}{2} I_3 + o(||v||^2) \end{aligned} \quad (12)$$

Estimating I_1 and considering the value of t_0 found on (3.2) we have:

$$I_1 \leq \left(1 - \frac{\tau+1}{2} \frac{||v||^2}{\gamma_n |S^{n-1}|} \right) h(0) |S^{n-1}| (1 - k_1 |\tilde{q}|^\alpha - k_1 \lambda^\alpha) + o(||v||^2) \quad (13)$$

For some constant k_1 .

Now for I_2 we use the orthogonality between v and u_q^τ (see lemma (3.3)), lemma (3.4), and the fact $T_\varphi u_q$ is bounded, we reach:

$$\begin{aligned} I_2 &= \int_{S^{n-1}} \bar{h}(x) u_q^\tau v d\sigma = \int_{S^{n-1}} \bar{h}(x) u_q^\tau v d\sigma - m \int_{S^{n-1}} u_q^\tau v d\sigma \\ &= \int_{B_{2\rho}(0)} (\bar{h}(x) - m) u_q^\tau v d\sigma \leq C \rho_0^\alpha ||v|| \leq C_4 ||v|| (|\tilde{q}|^\alpha + \lambda^\alpha) \end{aligned} \quad (14)$$

In order to estimate I_3 we consider that over a n -dimensional manifold M with boundary, the eigenvalues of the laplacian operator holds the inequality $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \dots \lambda_n$, where the first no zero eigenvalue can be variationally seen as:

$$\lambda_2 = \inf \left\{ \frac{\int_{B^n} |\nabla u|^2 dv}{\int_{S^{n-1}} u^2 d\sigma} : f \in H_1^2(B^n) - \{0\} \right\}$$

Our first no zero eigenvalue is $\lambda = 1$ and as $T_\varphi v$ is orthogonal to the coordinate functions and constants (see (2.8)), then for some $c > 0$:

$$1 + c \leq \frac{\int_{B^n} |\nabla T_\varphi v|^2 dv}{\int_{S^{n-1}} (T_\varphi v)^2 d\sigma}.$$

Adding γ_n both sides we have:

$$1 + c + \gamma_n \leq \frac{E(T_\varphi v)}{\int_{S^{n-1}} (T_\varphi v)^2 d\sigma}.$$

Now as $E(T_\varphi v) = E(v)$ and:

$$\|v\|^2 = \|T_\varphi v\|^2 \geq (\gamma_n + 1 + c) \int_{S^{n-1}} (T_\varphi v)^2 d\sigma$$

On the other hand:

$$\begin{aligned} \int_{S^{n-1}} u_q^{\tau-1} v^2 d\sigma &= \int_{S^{n-1}} u_q^{\tau-1} (\varphi(x)) v^2 (\varphi(x)) \det(d\varphi(x)) d\sigma \\ &= \int_{S^{n-1}} \frac{(T_\varphi u_q)^{\tau-1}}{\det(d\varphi(x))^{\frac{n-2}{2(n-1)}(\tau-1)}} \frac{(T_\varphi v)^2}{\det(d\varphi(x))^{\frac{n-2}{2(n-1)}2}} \det(d\varphi(x)) d\sigma \\ &= \int_{S^{n-1}} (T_\varphi v)^2 d\sigma \end{aligned}$$

Hence.

$$I_3 \leq h(0) \int_{S^{n-1}} u_q^{\tau-1} v^2 d\sigma = h(0) \int_{S^{n-1}} (T_\varphi v)^2 d\sigma \leq \frac{h(0)}{\gamma_n + 1 + c} \|v\|^2. \quad (15)$$

Now using (13), (14) and (15) in (12), exists $\beta > 0$ such as:

$$\bar{J}_\tau(u) \leq h(0)|S^{n-1}| [1 - \beta(|\tilde{q}|^\alpha + \lambda^\alpha + \|v\|^2)] \quad (16)$$

then.

$$\begin{aligned} J_p(u) &\leq |\bar{J}_p(u) - \bar{J}_\tau(u)| + |J_p(u) - \bar{J}_p(u)| + \bar{J}_\tau(u) \\ &\leq o_p(1) + C_3 \lambda^{(n-1)-(n-2)\delta_p} + C_3 \|v\|^{p+1} + h(0)|S^{n-1}| [1 - \beta(|q|^\alpha + \lambda^\alpha + \|v\|^2)] \end{aligned}$$

As $p \rightarrow \tau$ we obtain the result. ■

4 The variational scheme

In this chapter we show the solution to the system of equations (6) for each $p < \tau$. We previously defined:

$$S = \left\{ u \in H_1^2(B^n) : \|u\|^2 = \int_{B^n} |\nabla u|^2 dv + \int_{S^{n-1}} \gamma_n u^2 d\sigma = \gamma_n |S^{n-1}|, u \geq 0 \right\}$$

S is a closed set and it can be proved that the functional $J_p(u) = \int_{S^{n-1}} h u^{p+1} d\sigma$ is a Lipschitz Continuous, Compact operator and hence if $(u_i)_{i \in \mathbb{N}}$ is a sequence in $S \subset H_1^2(B^n)$ such that:

$$J'_p(u_i)|_{T_{u_i}(S)} \rightarrow 0 \quad \text{when } i \rightarrow \infty \quad (17)$$

Then $(u_i)_{i \in \mathbb{N}}$ has a convergent subsequence in S .

4.1 Mountain Pass Proof of the Main Result

By hypothesis, h has at least two positive local maximums, let r_1 and r_2 , the smaller positive local maximum of h . By propositions (3.1) and (3.5), there are two disjoint open sets $\overset{\circ}{\Sigma}_1, \overset{\circ}{\Sigma}_2 \subset S$, $\psi_i \in \overset{\circ}{\Sigma}_i$, $p_0 < \tau$ and $\delta > 0$ such as for all $p \geq p_0$:

$$J(\psi_i) > h(r_i)|S^{n-1}| - \frac{\delta}{2}, \quad i = 1, 2;$$

and

$$J(u) \leq h(r_i)|S^{n-1}| - \delta, \forall u \in \partial\Sigma, \quad i = 1, 2; \quad (18)$$

Let γ a pathway in S that ties ψ_1 and ψ_2 . We define the pathway family:

$$\Gamma = \{\gamma \in C([0, 1], S) : \gamma(0) = \psi_1, \gamma(1) = \psi_2\}$$

Now lets define:

$$c_p = \sup_{\gamma \in \Gamma} \min_{u \in \gamma} J_p(u).$$

By the Mountain Pass Theorem, there is a critical u_p of J_p in S such as:

$$J_p(u_p) = c_p$$

Besides as consequence of (18) and the definition of c_p , we have:

$$J_p(u_p) \leq \min_i h(r_i)|S^{n-1}| - \delta.$$

Now, as we have showed the existence of a critical point u_p of J_p in S , by lem (2.1), then we have there is a solution of the equation system (5) ■

5 A priori Estimates

In the last section, we proved the existence of a positive solution u_p to the subcritical equation (5) for each $p < \tau$. Now we prove that as $p \rightarrow \tau$, there is a subsequence of $\{u_p\}$, which converge to a solution u_0 of (1). In virtue to the Arzela-Ascoli Theorem convergence is based on the following estimate.

Theorem 5.1. *Assume h Satisfies the flatness condition, then there is a $p_0 < \tau$, such that for all $p_0 < p < \tau$ the solutions of (5) obtained in the variational scheme are uniformly bounded.*

To prove the theorem we estimate the solutions on three regions, h negative and away from zero, h positive and away from zero and h close to zero.

5.1 h Negative and away from zero

In this section we derive a priori estimates in the region where h is negative and bounded away from zero for all positive solutions of (1). We show the proof for the critical exponent $\tau = \frac{n}{n-2}$, however with a result we can extend this to $1 < p < \tau$. Now we will start introducing several established results that are necessary to bound the solution of (5).

Lemma 5.2. *Let $w \in C^2(\mathbb{R}_+^n) \cap C^1(\partial\mathbb{R}_+^n)$ a non negative function, B_1^+ the unit ball in \mathbb{R}_+^n and $C(x)$ a bounded function on \mathbb{R}_+^n . If $w \geq 0$ in $\partial'' B_1^+$ and satisfies:*

$$\begin{cases} \Delta_g w = 0 & \text{on } B_1^+, \\ \frac{\partial w}{\partial \eta} - C(x)w \geq 0 & \text{on } \partial' B_1^+ = \partial B_1^+ \cap \partial\mathbb{R}_+^n, \end{cases} \quad (19)$$

Where $\frac{\partial w}{\partial \eta} - C(x)w$ is not equal zero on all $\partial' B_1^+$; then $w > 0$ in B_1^+ .

The proof of this result are the same steps for the proof of **lemma 2.1** in [11].

Lemma 5.3.

Proposition 5.4. *The solutions of (1) are uniformly bounded in the region where $h(x) \leq -\delta$, for every $\delta > 0$. The bound depends on δ , $\text{dist}(\{x|h(x) \leq -\delta\}, S_0)$, and the lower bound of h , where $S_0 = \{x|h(x) = 0\}$.*

To prove this proposition we first need to show that, for any given solution of (1), the values of u at two given points x_0, x_1 are comparable, in other words $u(x_0) \leq Cu(x_1)$.

Lemma 5.5. *Let x_0 be a point where h is negative. Let $3\epsilon_0 = \text{dist}(x_0, S_0)$ and*

$$h(x) \leq -\delta_0 \quad \text{for all } x \in B_{2\epsilon_0}(x_0).$$

Assume that $h(x) \geq -M$ for all $x \in \mathbb{R}^n$. Then there exists a constant $C = C(\epsilon_0, \delta_0, M)$, such that for any point x_1 we have:

$$u(x_0) \leq C(|x_1 - x_0| + \epsilon_0)^{n-2} u(x_1)$$

Proof.

For convenience we make take an extension of the stereographic projection from B^n into \mathbb{R}_+^n , then for the equation system (1) is equivalent to consider the following equation system on \mathbb{R}_+^n :

$$\left\{ \begin{array}{ll} \Delta u = 0 & \text{in } \mathbb{R}_+^n, \\ \frac{\partial u}{\partial \eta} = h u^{n/n-2} & \text{over } \partial \mathbb{R}_+^n, \end{array} \right. \quad (20)$$

Now we will use the Kelvin Transform and the Maximum Principle to prove the result. Lets make a translation of coordinates, sothat the point \bar{x} becomes the origin. The Laplacian operator is invariant under such translation so ti can be done. Then we make a rescaling. Let:

$$u_0(x) = \epsilon_0^{\frac{n-2}{2}} u(\epsilon_0 x)$$

A simple calculation shows that u_0 satisfies:

$$\left\{ \begin{array}{ll} \Delta u_0 = 0 & \text{in } \mathbb{R}_+^n, \\ \frac{\partial u_0}{\partial \eta} = \bar{h} u_0^{n/n-2} & \text{over } \partial \mathbb{R}_+^n, \end{array} \right. \quad (21)$$

where $\bar{h}(x) = h(\epsilon_0 x)$.

Now lets take $v(x) = \frac{1}{|x|^{n-2}} u_0(\frac{x}{|x|^2})$ the Kelvin Transform of u_0 , A direct calculation proofs that v holds the equation system:

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}_+^n, \\ \frac{\partial v}{\partial \eta} = \bar{h} \left(\frac{x}{|x|^2} \right) v^{n/n-2} & \text{over } \partial \mathbb{R}_+^n, \end{cases} \quad (22)$$

Now lets compare the function $\alpha v(x)$ with u_o in the unit ball $B_1^+(0)$. To this end lets define

$$w(x) = \alpha v(x) - u_o(x).$$

Then $w(x)$ satisfies the equation system:

$$\begin{cases} \Delta w = 0 & \text{in } B_1^+(0), \\ \frac{\partial w}{\partial \eta} = \alpha \bar{h} \left(\frac{x}{|x|^2} \right) v^{n/n-2} - \bar{h}(x) u_o^{n/n-2} > 0 & \text{over } \partial' B_1^+(0) = \partial B_1^+(0) \cap \partial \mathbb{R}_+^n, \end{cases} \quad (23)$$

As a direct calculation we get $\Delta w = 0$, now for the boundary term lets take $x \in B_1^+(0)$

$$\begin{aligned} \frac{\partial w}{\partial \eta} &= \alpha \bar{h} \left(\frac{x}{|x|^2} \right) v^{n/n-2} - \bar{h}(x) u_o^{n/n-2} \\ &= \alpha \bar{h} \left(\frac{x}{|x|^2} \right) v^{n/n-2} - \alpha^{\frac{-2}{n-2}} \bar{h} \left(\frac{x}{|x|^2} \right) u_o^{n/n-2} + \alpha^{\frac{-2}{n-2}} \bar{h} \left(\frac{x}{|x|^2} \right) u_o^{n/n-2} - \bar{h}(x) u_o^{n/n-2} \end{aligned}$$

and

$$\frac{\partial w}{\partial \eta} - \alpha^{\frac{-2}{n-2}} \bar{h} \left(\frac{x}{|x|^2} \right) \left[(\alpha v)^{n/n-2} - u_o^{n/n-2} \right] = \left(\alpha^{\frac{-2}{n-2}} \bar{h} \left(\frac{x}{|x|^2} \right) - \bar{h}(x) \right) u_o^{n/n-2}$$

Now by the Mean value theorem there is a continuous function ϕ valued between αv and u such as:

$$\frac{\partial w}{\partial \eta} - \alpha^{\frac{-2}{n-2}} \bar{h} \left(\frac{x}{|x|^2} \right) \phi w = \left(\alpha^{\frac{-2}{n-2}} \bar{h} \left(\frac{x}{|x|^2} \right) - \bar{h}(x) \right) u_0^{n/n-2}$$

Moreover since $-M \leq h(x) \leq -\delta$ then:

$$\bar{h}(x) - \alpha^{\frac{-2}{n-2}} \bar{h} \left(\frac{x}{|x|^2} \right) \leq (-\delta_0 + M\alpha^{\frac{-2}{n-2}})$$

With α sufficiently large we have $(-\delta_0 + M\alpha^{\frac{-2}{n-2}}) < 0$ and

$$\frac{\partial w}{\partial \eta} - \alpha^{\frac{-2}{n-2}} \bar{h} \left(\frac{x}{|x|^2} \right) \phi w = \left(\alpha^{\frac{-2}{n-2}} \bar{h} \left(\frac{x}{|x|^2} \right) - \bar{h}(x) \right) u_0^{n/n-2} > 0$$

We can conclude that $\frac{\partial w}{\partial \eta} > 0$ in $\partial' B_1^+(0)$ using maximum principle and Hopf lemma we show that the minimum of w belongs to $\partial'' B_1^+(0) = \partial B_1^+(0) \cap \mathbb{R}_+^n$ as $w \geq 0$ in $\partial'' B_1^+(0)$ then $w \geq 0$ in $B_1^+(0)$. Thus

$$u_0(x) \leq \alpha v(x) = \alpha \frac{1}{|x|^{n-2}} u_0 \left(\frac{x}{|x|^2} \right)$$

$$\epsilon_0^{\frac{n-2}{2}} u(\epsilon_0 x) \leq \frac{\epsilon_0^{\frac{n-2}{2}}}{|x|^{n-2}} u \left(\frac{\epsilon_0 x}{|x|^2} \right)$$

and

$$u(x) \leq \frac{\epsilon_0^{n-2}}{|x|^{n-2}} u \left(\frac{\epsilon_0^2 x}{|x|^2} \right)$$

Thus for $x \in B_{\epsilon_0}(\bar{x})$ with α who depends only on δ_0 and M we have.

$$u(x) \leq \frac{\epsilon_0^{n-2}}{|x - \bar{x}|^{n-2}} u \left(\epsilon_0^2 \frac{x - \bar{x}}{|x - \bar{x}|^2} + \bar{x} \right)$$

Now given x_1 and x_0 lets take a point \bar{x} , such as those points are on the same line with x_0 in between and satisfy.

$$|x_0 - \bar{x}| |x_1 - \bar{x}| = \epsilon_0^2$$

Now as the vector $x_1 - \bar{x}$ has the same direction as $x_0 - \bar{x}$ then.

$$x_1 - \bar{x} = \frac{\epsilon_0^2}{|x_0 - \bar{x}|^2}(x_0 - \bar{x})$$

Hence.

$$u(x_0) \leq \alpha \frac{\epsilon_0^{n-2} |x_1 - \bar{x}|^{n-2}}{\epsilon_0^{2n-4}} u \left(\epsilon_0^2 \frac{x_0 - \bar{x}}{|x_0 - \bar{x}|^2} + \bar{x} \right) \leq \frac{\alpha}{\epsilon^{n-2}} [|x_1 - x_0| + |x_0 - \bar{x}|]^{n-2} u(x_1)$$

Thus,

$$u(x_0) \leq C [|x_1 - x_0| + \epsilon]^{n-2} u(x_1)$$

where C only depends on δ_0 , ϵ_0 and M . If we take the case $2 < n < \frac{2p}{p-1}$ the inequality holds for $1 \leq p \leq \frac{n}{n-2}$.

■

Now lets Prove the proposition 5.4.

Proof.

As the energy functional is bounded for solutions $u_p \in S$ then for $B_{\epsilon_0}(x_0) \subset S^{n-1}$

$$\int_{B_{\epsilon_0}(x_0)} u^p d\sigma \leq \gamma_n |S^{n-1}|$$

Applying lemma 5.5 and like the infimum is equal or less than the average of the function.

$$\begin{aligned} u(x) &\leq \inf_{\bar{x} \in B_{\epsilon_0}(x_0)} u(\bar{x}) \leq \frac{C}{|B_{\epsilon_0}(x_0)|} \int_{B_{\epsilon_0}(x_0)} u d\sigma \\ &\leq \frac{C}{|B_{\epsilon_0}(x_0)|} \left(\int_{B_{\epsilon_0}(x_0)} u^p d\sigma \right)^{\frac{1}{p}} |B_{\epsilon_0}(x_0)|^{1-\frac{1}{p}} \leq K_2 \end{aligned}$$

Where K_2 only depends on δ_0 , $\text{dist}(\{x|h(x) \leq -\delta_0\}, \{x|h(x) = 0\})$ and the inferior bound of h , Hence the solutions are uniformly bounded when h negative.

■

5.2 h Small and close to zero

The bound of the solution here is consequence mainly due to the energy functional bound, for rotationally symmetric h on ∂B^n we use some blow up analysis near $h(x) = 0$, also as B^n is compact the maximum for u_p should be on the boundary S^{n-1} then we will consider any blow up point there.

Proposition 5.6. *Let $\{u_p\}$ be the solutions of the subcritical equation (5) obtained by the variational approach, there exists a $p_0 < \tau$ and $\delta > 0$, such that for all $p_0 < p \leq \tau$, $\{u_p\}$ are uniformly bounded in the regions where $|h(x)| \leq \delta$.*

Proof.

First let's see that the energy functional is bounded on those solutions, indeed as $|h| < \delta$ for some $\delta > 0$, if u_p is a solution of (5) then there is $\lambda \in \mathbb{R}$ and $w_p \in S$ such as $u_p = \lambda w_p$, then.

$$E(u_p) = \int_{S^{n-1}} h u_p^{p+1} d\sigma = \lambda^{p+1} \int_{S^{n-1}} h w_p^{p+1} d\sigma \leq C \int_{S^{n-1}} w_p^{p+1} d\sigma \leq C_2 |S^{n-1}|$$

Now let's argue by contradiction, let's suppose that there is a subsequence $\{u_i\}$ with $u_i = u_{p_i}$, $p_i \rightarrow \tau$, and a sequence of points $\{x_i\}$, with $x_i \rightarrow x_0$ $u_i(x_i) \rightarrow \infty$ Now we will also take $h(x_0) = 0$ as if we take x_0 as explosion point with $h(x_0) = \delta_1$ we just need to take $\delta_2 < \delta_1$, and avoid the explosion point.

Now we will use a rescaling argument to reach a contradiction, As x_i may not be a local maximum of u_i , we choose a point near x_i , which is almost a local maximum.

Let K be any large number and let.

$$r_i = 2K [u_i(x_i)]^{-\frac{p_i-1}{2}}$$

In a small neighborhood of x_0 , choose a local coordinate and let.

$$s_i(x_i) = u_i(x_i) (r_i - |x - x_i|)^{\frac{2}{p_i-1}}$$

Let a_i be the maximum of $s_i(x)$ in $B_{r_i}(x_i)$. Let $\lambda_i = [u_i(a_i)]^{-\frac{p_i-1}{2}}$, then from the definition of a_i we have:

1. $B_{\lambda_i K}(a_i) \subset B_{r_i}(x_i)$
2. $u_i(x) \leq cu_i(a_i)$ in $B_{\lambda_i K}(a_i)$

Indeed, lets take $x \in B_{\lambda_i K}(a_i)$, now as

$$\lambda_i K = \frac{r_i}{2} \left[\frac{u_i(x_i)}{u_i(a_i)} \right]^{\frac{p_i-1}{2}} = \frac{r_i}{2} \left[\frac{r_i - |x - x_i|}{r_i} \right] \left[\frac{s_i(x_i)}{s_i(a_i)} \right]^{\frac{p_i-1}{2}} < \frac{r_i - |x - x_i|}{2}$$

then

$$|x - x_i| \leq |x - a_i| + |a_i - x_i| < \lambda_i K + |a_i - x_i| < \frac{r_i}{2} + \frac{|a_i - x_i|}{2} < r_i$$

then $x \in B_{r_i}(x_i)$ and $B_{\lambda_i K}(a_i) \subset B_{r_i}(x_i)$.

From last inequality we also have.

$$r_i - |x - x_i| > \frac{r_i}{2} - \frac{|a_i - x_i|}{2}$$

Thus

$$u_i(x) (r_i - |x - x_i|)^{\frac{2}{p_i-1}} > u_i(x) \left(\frac{r_i}{2} - \frac{|a_i - x_i|}{2} \right)^{\frac{2}{p_i-1}}$$

And as $s_i(x) \leq s_i(a_i)$ in $B_{\lambda_i K}(a_i) \subset B_{r_i}(x_i)$ then

$$u_i(x) < 2^{\frac{2}{p_i-1}} u_i(a_i) < C u_i(a_i)$$

For all $x \in B_{\lambda_i K}(a_i)$. Now we can make a rescaling.

$$v_i(x) = \frac{1}{u_i(a_i)} u_i(\lambda_i x + a_i)$$

From the fact $u_i(x) \leq cu_i(a_i)$ in $B_{\lambda_i K}(a_i)$ we have $v_i(x)$ bounded on $B_K(0)$ and $v_i(0) = 1$ for all i . Now as consequence from the mean value inequality

we have $v_i(x)$ is equicontinuous on $B_K(0)$, then by Arzela-Ascoli Theorem $\{v_i\}$ has a subsequence that uniformly converges to a harmonic function v_0 in the closure of $B_K(0) \subset \mathbb{R}^n$ and $v_0(0) = 1$.

Moreover, due to $|v_i(x) - v_0(x)| \leq ||v_i(x) - v_0(x)|| < \epsilon$ for $i \geq N$ for some big N . we have $|v_0| \leq |v_i| + |v_0 - v_i|$ thus $|v_0|^{\tau+1} \leq 2^{\tau+1} |v_i|^{\tau+1} + |v_0 - v_i|^{\tau+1}$ then.

$$\begin{aligned} \int_{B_K(0)} v_i^{\tau+1}(x) dx &\geq \frac{1}{2^{\tau+1}} \int_{B_K(0)} v_0^{\tau+1}(x) dx - \int_{B_K(0)} |v_0(x) - v_i(x)|^{\tau+1} dx \\ &\geq \frac{C_1}{2^{\tau+1}} \int_{B_K(0)} v_0(x) dx - \epsilon^{\tau+1} |B_K(0)| \geq (C_2 - \epsilon^{\tau+1}) |B_K(0)| \end{aligned}$$

Last inequality as $h(x_0) = 0$ then u satisfies:

$$\begin{cases} \Delta_g v_0 = 0 & \text{on } B_K(0) \subset \mathbb{R}_+^n, \\ \frac{\partial v_0}{\partial \eta} = 0 & \text{on } \partial B_K(0) \subset \partial R_+^n, \end{cases} \quad (24)$$

As $\frac{\partial v_0}{\partial \eta} = 0$ we can take a reflection of v_0 , let this function be $\hat{v}_0 : B_K(0) \subset \mathbb{R}_-^n$ and let's define

$$w(x) = \begin{cases} v_0(x) & \text{if } x \in \mathbb{R}_+^n, \\ v_0(\hat{x}) & \text{if } x \in \mathbb{R}_-^n, \end{cases} \quad (25)$$

$w(x)$ is then an harmonic function in \mathbb{R}^n and by the mean value theorem we have $w(y) = \frac{1}{|B_K(y)|} \int_{B_K(y)} w dx$ and $v_0(y) = \frac{1}{2|B_K(y)|} \int_{B_K(y)} v_0 dx$, in particular take $y = 0$ and $\epsilon < C_2^{-\tau-2}$.

Then for i big enough we have.

$$\int_{B_K(0)} v_i^{\tau+1}(x) dx \geq cK^n \quad (26)$$

For some positive constant c . on the other hand, from the boundedness of the energy $E(u_i)$ we have.

$$\int_{S^n} u_i^{\tau+1} dV \leq C$$

Ahora para cualquier $K > 0$ tengo que.

$$\int_{S^n} u_i^{\tau+1} dV = \int_{\mathbb{R}^n} u_i^{\tau+1}(\pi(x)) dx \geq \frac{1}{u_i^{\tau+1}(a_i)} \int_{B_K(0)} u_i^{\tau+1}(x) dx \geq \int_{B_K(0)} v_i^{\tau+1}(x) dx$$

Now as $B_K(0) \subset \mathbb{R}_+^n$ and with π as the inverse of the extension of the stereographic projection to the ball, we reach a contradiction, because

$$C \geq \int_{S^n} u_i^{\tau+1} dV \geq \int_{B_K(0)} v_i^{\tau+1}(x) dx \geq cK^n$$

but this inequality doesn't hold for K large. Then u_i must be uniformly bounded.

■

5.3 h Positive and away from zero

Proposition 5.7. *Let $\{u_p\}$ be solutions of the subcritical problem (5) obtained by the variational aproach. Then there exists a $p_0 < \tau$. Such that for all $p_0 < p < \tau$ and for any $\delta > 0$, $\{u_p\}$ are uniformly bounded in the regions where $h(x) \geq \delta$.*

proof

The argument starts being the same as for $|h| < \delta$, Let $\{x_i\}$ be sequence of points such that $u_i(x_i) \rightarrow \infty$ and $x_i \rightarrow x_0$ con $h(x_0) > 0$. Let $r_i(x), s_i(x)$ and $v_i(x)$ be defined as for the case $|h| < \delta$ and in the same fashion $v_i(x)$ converges to standard function $v_0(x)$ in \mathbb{R}_+^n with

$$\begin{cases} -\Delta v_0 = 0 & \text{in } \mathbb{R}_+^n, \\ \frac{\partial v_0}{\partial \eta} = hu^{n/n-2} & \text{over } \partial\mathbb{R}_+^n, \end{cases} \quad (27)$$

It follows that

$$\int_{B_{r_i}(x_0)} u^{\tau+1} dV \geq c_0 > 0.$$

Because the total energy of u_i is bounded, we can only have finitely many such x_0 . Hence $\{u_i\}$ has finite isolated blow up points. As consequence of this result, in [7] (proposition 4.11) we have:

Theorem 5.8. *Let u_i be a solution of (5) for $n \geq 3$, Assume that for each critical point x_0 we have the flatness condition of 1.1 the u_i can have at most one simple blow up point, and this point must be a local maximum of h . Besides $\{u_i\}$ behaves almost like a family of the standard functions u_q .*

However by the variational scheme, even one point of blow up is not possible. Lets take u_i the sequence of critical of points of the functional J_p obtained in the variational scheme. from the proof of proposition 18 we can obtain.

$$J_\tau(u_i) \leq \min_k h(r_k) |S^{n-1}| - \delta$$

for all the positive local maxima r_k of h . Now if $\{u_i\}$ blow up at x_0 , then by teor 5.8 we have.

$$J_\tau(u_i) \longrightarrow h(x_0) |S^{n-1}|$$

This is a contradiction. and proves proposition 5.7.

We can conclude that the sequence u_i is uniformly bounded and by Arzela-Ascoli Theorem has a subsequence converging to a solution of (1). This finish covering all cases for the proof of theorem 5.1 and complete the proof of 1.1.

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